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To solve this in integers, set $a + g = mq$, $h + b = nq$, where m and n are relatively prime. Then $a - g = np$, $h - b = mp$. Thus

$$(14)\quad a = \tfrac{1}{2}(mq + np),\quad g = \tfrac{1}{2}(mq - np),\quad h = \tfrac{1}{2}(nq + mp),\quad b = \tfrac{1}{2}(nq - mp).$$

Since m and n are relative prime, these four numbers are integers only in the following cases: mn odd, p and q both even or both odd; just one of m and n even, p and q even. *All integral solutions of* (13) *are given by* (14) *in which* m *and* n *are relatively prime integers, while* p *and* q *are both even or* p, q, m, n *are all odd.*

The last problem is evidently the same as that of finding all triangles whose three sides and one median are rational.

9. **Conclusion.** All of the problems mentioned in this paper have been completely solved except that of a general quadrilateral whose sides and diagonals (and area) are rational. That question reduces to the problem of making a quartic function equal to a rational square. To this same problem may be reduced the solution of various questions[1] relating to triangles and quadrilaterals, as well as many questions in Diophantine analysis. A complete solution of this common outstanding problem is much to be desired.

THE TRIANGLE OF REFERENCE IN ELEMENTARY ANALYTIC GEOMETRY.

By LENNIE PHOEBE COPELAND, Wellesley College.

While the use of the triangle of reference and homogeneous coördinates is common in advanced work in mathematics, comparatively little is done along this line by undergraduates. It seems possible that younger students might find it profitable and interesting to note the behavior and shape, especially at infinity, of some of the well-known curves, when plotted on a triangle of reference. This triangle may be explained very simply,[2] since it is formed by the three lines of reference (Fig. 1) CA, CB and AB or $y = 0$, $x = 0$ and $z = 0$ corresponding respectively to the X and Y axes of the Cartesian system and the "conventional line at infinity." The distances from these lines, numbers proportional to them, or to arbitrary multiples of them, determine the coördinates of any point. The selection of the negative and positive sides of the lines may be made arbitrarily. However, it is generally more convenient to determine them in such a manner that the coördinates of points within the triangle shall be positive. This region corresponds to the first quadrant determined by the Cartesian axes, those portions of the plane outside the triangle corresponding in pairs to the second, third and fourth quadrants. The coördinates of the vertices are $A(1, 0, 0)$, $B(0, 1, 0)$ and C $(0, 0, 1)$. The points of CB are characterized by the fact that x equals zero, and therefore any point will have the coördinates $(0, y, z)$; similarly points on CA and AB have coördinates $(x, 0, z)$ and $(x, y, 0)$ respectively. In general if P is any point not on a side of the triangle, there exist three numbers x, y, z all different from zero such that the projections of P from the vertices on the opposite sides have the coördinates $(0, y, z)$, $(x, 0, z)$ and $(x, y, 0)$. These numbers are known as the trilinear or homogeneous coördinates of P. When the coördinates are taken proportional to the distances themselves of a point from the sides of the triangle (the arbitrary multipliers equal) the bisectors of the angles of the triangle are the lines $y - z = 0$, $x - z = 0$ and $x - y = 0$, having the relation of harmonic conjugates to the corresponding bisectors of the exterior angles, the lines $y + z = 0$, $x + z = 0$ and $x + y = 0$. The former intersect the opposite sides of the triangle in the points D $(0, 1, 1)$, E $(1, 0, 1)$ and F $(1, 1, 0)$ respectively and meet in the point $P(1, 1, 1)$; the latter intersect the opposite sides produced in three points which lie on the line $x + y + z = 0$. In general any line through C has for its equation $y \pm mx = 0$ and is the harmonic conjugate of the line $y \mp mx = 0$. Similar equations hold for lines through B and A.

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[1] Dickson, *History of the Theory of Numbers*, vol. 2, 1920, pp. 165–224, 497.

[2] G. Salmon, *Treatise on Conic Sections*, fifth edition, London, 1869, chapter 4.

G. Salmon, *Higher Plane Curves*, second edition, Dublin, 1873, chapter 1.

C. A. Scott, *Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry*, London, 1894.

Clebsch-Lindemann, *Vorlesungen über Geometrie*, 2. Auflage, vol. 1, part 1, Leipzig, 1906, p. 119.

O. Veblen and J. W. Young, *Projective Geometry*, vol. 1, Boston, 1910, p. 174.

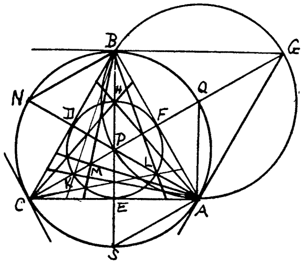

FIG. 1.

The line $x + y + z = 0$ becomes the line at infinity when the triangle of reference is equilateral. Hence it follows in this case that any curve[1] which cuts or touches the line $x + y + 1 = 0$ when referred to Cartesian axes will have infinite branches or points, when referred to this triangle. All lines intersecting on it then become parallel. Thus $NB : 2x + z = 0$ and $SA : 2y + z = 0$ are both parallel to $CF : x - y = 0$. Then if M is the midpoint of CF, the lines BM and AM are the harmonic conjugates of NB and SA or $2x - z = 0$ and $2y - z = 0$. In a similar manner the equations of other lines may be determined. With these facts in mind it is an easy matter to plot the ordinary curves, and although they appear somewhat disguised and distorted it is interesting to examine them from this new point of view. Moreover it is fascinating to see apparently simple functions don unsuspected singularities, while others break in two at unexpected places or appear with their separate branches united. For simplicity we may consider the coördinates as proportional to the distances themselves and the triangle of reference equilateral in the following examples.

The parabola $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$ which lies wholly in the first quadrant and is tangent to the x and y axes becomes the inscribed circle.

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[1] The term curve as here used means the graph of an equation. Strictly speaking a given equation represents one curve when plotted on a certain set of axes and a different curve when plotted on others. Also a given curve when referred to different systems of axes does not change although its equation does.

For, writing the equation in homogeneous coördinates

$$x^2 - 2xy + y^2 - 2xz - 2yz + z^2 = 0,$$

it is seen that if $x = 0$, then $(y - z)^2 = 0$, and therefore the line $x = 0$ meets the curve twice at its point of intersection with the line $y - z = 0$, or in other words (Fig. 1) CB is tangent to the curve at D. Likewise CA and AB are shown to be tangents at the points E and F respectively. Further if $x = z$, then $y(y - 4z) = 0$, and therefore the line $x - z = 0$ cuts the curve in two points, namely the points where $x - z = 0$ intersects the lines $y = 0$ and $y - 4z = 0$. Hence BE cuts the curve at E and H, likewise CF cuts at K and F, and AD at D and L. By similar methods an infinite number of pairs of corresponding points within the triangle can be determined. Now the lines $x + my = 0$, $x + mz = 0$ and $y + mz = 0$ (when m is positive) all intersect the curve in imaginary points. Therefore no part of it lies without the triangle, and the resulting curve can be proved a circle. Here then we see that the two finite tangents $x = 0$ and $y = 0$ remain such, but the tangent at infinity having been brought into the plane is also visible.

The circumscribed circle is obtained by plotting the hyperbola $xy + x + y = 0$ which has a tangent at the origin $x + y = 0$ and the asymptotes $x + 1 = 0$ and $y + 1 = 0$. From its homogeneous equation it is seen that if $x = 0$, then $yz = 0$, or CB cuts the curve at C and B, and similarly CA cuts at C and A, AB at A and B. If $x = z$, then $z(2y + z) = 0$. Hence BE cuts at B and S, and similarly AD at A and N, CF at C and Q. The lines $x + y = 0$, $x + z = 0$ and $y + z = 0$ are tangents at the points C, B and A respectively. Thus the finite tangent is preserved, the two asymptotes become as was to be expected tangents at the points of intersection with $z = 0$, the line at infinity, and the two separate branches of the given hyperbola unite in an unbroken curve, namely a circle.

If we consider the hyperbola $xy = 1$, we see (Fig. 1) that the asymptote $x = 0$ is tangent to the curve at B when plotted on the triangle of reference. Likewise $y = 0$, the second asymptote, becomes a visible tangent at A. $x - y = 0$ cuts the curve at its points of intersection with $x - z = 0$ and $x + z = 0$, that is at P $(1, 1, 1)$ and G $(1, 1, -1)$. Other points obtained in a similar manner show the resulting curve to be a circle, the portion APB corresponding to that portion of the hyperbola in the first quadrant and AGB to that in the third. It is interesting to note that the parabolas $y^2 = x$ and $x^2 = y$ when referred to this triangle become equal circles through P having CA and CB, respectively, for chords.

Now let us consider the hyperbola $4xy = 1$ which is tangent to the line $x + y + 1 = 0$. From its homogeneous equation we observe (Fig. 2) that $x = 0$ and $y = 0$ are tangents to the curve at B and A respectively, while $z = 0$ cuts it at these same points; $x + y = 0$ in imaginary points; $x - y = 0$ at the points where it intersects the lines $2y - z = 0$ and $2y + z = 0$, namely M and infinity; $x - z = 0$ at the points where it intersects the lines $z = 0$ and $4y - z = 0$, namely B and T; likewise $y - z = 0$ at A and R; and in general if m is positive

$x + my = 0$ cuts the curve in imaginary points; $x - my = 0$ in two harmonic points one inside the triangle and the other outside; $x + mz = 0$ in two points one of which is B and the other the point where $x + mz = 0$ meets $4my + z = 0$, etc. Hence the curve falling entirely within the regions corresponding to the first and third quadrants, with one point at infinity, takes the form of a parabola.

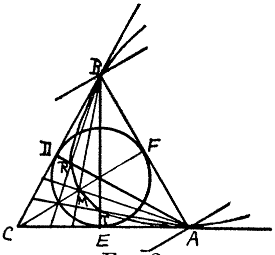

FIG. 2.

If the equation of the hyperbola is taken as $x^2 - y^2 = 1$ or $x^2 - y^2 = z^2$ then $x = 0$ cuts it in imaginary points; $y = 0$ at E and infinity; $z = 0$ at F and infinity; $y - mz = 0$ (m positive) at its points of intersection with the lines $x + z\sqrt{m^2 + 1} = 0$ and $x - z\sqrt{m^2 + 1} = 0$ giving an infinity of points inside the triangle and a corresponding infinity on the other side of the line CB, these points corresponding to those in the first and second quadrants respectively. The lines $y + mz = 0$ also cut the curve in a set of points outside the triangle in the regions corresponding to the third and fourth quadrants. The tangents of the original hyperbola $x - z = 0$ and $x + z = 0$ become a tangent at E and an asymptote respectively, while the former asymptotes $x - y = 0$ and $x + y = 0$ become one a tangent and the other an asymptote. Hence the resulting curve is a hyperbola united at one of the points where the original hyperbola broke, and breaking at one new point, namely where the line $x + y + 1 = 0$ cuts $x^2 - y^2 - 1 = 0$.

Similarly a circle about the origin in the ordinary system of coordinates referred to this triangle becomes a parabola, ellipse or hyperbola according as its radius equals, is less than or greater than, $\frac{1}{2}\sqrt{2}$.

If we refer the cubical parabola to an equilateral triangle of reference, we have in homogeneous coördinates $yz^2 - x^3 = 0$ (Fig. 3). Hence the line $x - z = 0$ cuts the curve twice at B and once at P; $x + z = 0$ twice at B and once at G: $x - y = 0$ at C, P and G: $x + y = 0$ at C; $y - z = 0$ at P; $y + z = 0$ at G; $x = 0$ twice at B and once at C; $y = 0$ three times at C, the point of inflexion. Likewise $z = 0$ cuts the curve three times at B, and as this is the only one of the lines through B ($z \pm mx = 0$) which has three intersections with the curve at this point, we see that the multiple point at B must be a cusp and the line at infinity $z = 0$ a cuspidal tangent.

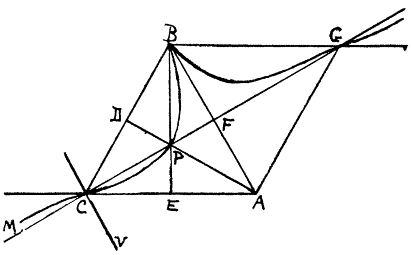

FIG. 3.

Other cubics and quartics such as the Witch of Agnesi, the Cissoid of Diocles, the Conchoid of Nicomedes and the tricuspidal quartic are interesting to plot on the triangle of reference. Moreover by varying the form of the triangle or the parameters of any given equation it may be shown that the resulting graph may assume any one of its projective forms. Thus metrical properties are changed but projective properties are preserved, and hence, in any case, the behavior at infinity may be noted.